Stabilizing the discrete vortex of topological charge $S=2$

P. G. Kevrekidis[1] and D. J. Frantzeskakis[2]
[1]*Department of Mathematics and Statistics, University of Massachusetts, Amherst Massachusetts 01003-4515, USA*
[2]*Department of Physics, University of Athens, Panepistimiopolis, Zografos, Athens 15784, Greece*
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We study the instability of the discrete vortex with topological charge $S=2$ in a prototypical lattice model and observe its mediation through the central lattice site. Motivated by this finding, we analyze the model with the central site being *inert*. We identify analytically and observe numerically the existence of a range of linearly stable discrete vortices with $S=2$ in the latter model. The range of stability is comparable to that of the recently observed experimentally $S=1$ discrete vortex, suggesting the potential for observation of such higher charge discrete vortices.

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I. INTRODUCTION

In the past decade, lattice systems described by differential-difference equations in which the evolution variable is continuum and the spatial variables are discrete have been a subject of increasing interest [1]. These systems appear in many diverse physical contexts, describing, e.g., the spatial dynamics of optical beams in coupled waveguide arrays in nonlinear optics [2], the temporal evolution of Bose-Einstein condensates (BECs) in optical lattices in soft-condensed matter physics [3], the DNA double strand in biophysics [4], and so on.

One of the principal directions of interest in these lattice systems consists of the effort to understand the features of their localized, solitary wave solutions. In two dimensions, such structures can be discrete solitons [5] or discrete vortices (i.e., structures that have topological charge over a discrete contour) [6]. In the past two years, there has been a considerable effort towards the observation of both entities in the context of optics, utilizing photorefractive crystals: regular discrete solitons, dipole solitons, soliton trains, soliton necklaces, and vector solitons were observed [7], while two groups were independently able to experimentally produce robust discrete vortex states [8]. On the other hand, experimental developments in the physics of BECs closely follow with prominent recent results, including the observation of bright, dark, and gap solitons in quasi-one-dimensional settings [9], and with the generation of similar structures in higher dimensions appearing within experimental reach [10].

The above discussed experimental realization of discrete vortices of topological charge $S=1$ [8] (i.e., with a 2π phase shift around a discrete contour) poses the question of whether higher topological charge discrete vortices could also potentially be experimentally realizable. The most natural higher topological charge state to consider is then the vortex with $S=2$. However, lattice computations with a prototypical discrete model, namely the the discrete nonlinear Schrödinger (DNLS) equation, had identified that mode to be *always* unstable [6]. In this work we will revisit this topic and examine in some detail the instability of the $S=2$ discrete vortex in the framework of the DNLS equation [11] which, in different variants, is relevant to all of the above mentioned, experimentally tractable settings. Our scope is to offer some insight on the nature of the instability, which will, in turn, allow us to suggest an explicit mechanism for its stabilization by means of the inclusion of an impurity at its center. We analyze the latter case in detail and establish (analytically and numerically) the stability of the $S=2$ vortex in that setting for parametric regimes similar to the ones for which the discrete vortex of $S=1$ has been found to be stable. We should note in passing that experiments with vortices in BECs have not been performed, to the best of our knowledge, in the presence of periodic optical lattice potentials despite the theoretical suggestion of interesting phenomena in the latter context [12]. However, vortices of higher topological charge [13] have been constructed (including giant vortices [14]) and a considerable amount of interest has revolved around their stabilization in theoretical work [15] as well as in earlier experimental work in superfluid He [16]. Since higher dimensional optical lattices are readily experimentally accessible (see, e.g., Refs. [3,10]), the phenomenology presented herein might be directly relevant to experimentally accessible settings involving matter waves. Notice also that an additional complication appears to be that most of the above experiments were conducted in repulsive condensates (with defocusing nonlinearities), while the present analysis will be relevant for focusing nonlinearities. However, as is discussed in detail in Refs. [17,18], it is at present very straightforward to modify the magnitude and sign of the scattering length of interatomic interactions (and hence of the nonlinearity coefficient) over a wide range. Therefore the attractive nature of the interactions for the vortices considered herein does not pose an obstacle to the experimental realization of the relevant setting.

Our presentation is structured as follows: in Sec. II, we give the theoretical analysis of the existence and stability of the $S=2$ vortex; in Sec. III, we complement the analysis with numerical results. Finally, in Sec. IV, we summarize our findings and present our conclusions.

II. ANALYTICAL RESULTS

We consider the DNLS equation

$$i\dot{u}_{n,m} = -\epsilon\Delta_2 u_{n,m} - |u_{n,m}|^2 u_{n,m}, \tag{1}$$

where u is the complex field (the envelope of the electric field in optics or the wave function in BECs), ϵ is the coupling constant (the "tunneling rate" between adjacent sites), while $\Delta_2 u_{n,m}=u_{n+1,m}+u_{n-1,m}+u_{n,m+1}+u_{n,m-1}-4u_{n,m}$ is the discrete Laplacian. We seek standing wave, localized solutions in the standard form $u_{n,m}=\phi_{n,m}e^{i(\mu-4\epsilon)t}$. Our approach is based on the Lyapunov-Schmidt theory for the existence of solutions [19] and on linear stability analysis for tracing the stability eigenvalues of the corresponding solutions, similarly to what was done for discrete solitons and vortices in Ref. [20]. Let us mention in passing that for quasilinear solutions in the presence of an external potential with appropriate spectrum, one can also use techniques similar to those of Ref. [21] to analyze the stability of higher charge vortices. However, given that we are interested in solutions even in the strongly nonlinear regime here, we do not use the latter approach.

Our starting point is the so-called anticontinuum limit of $\epsilon=0$ [22], where the nonlinear oscillators of our two-dimensional lattice are uncoupled. We excite a discrete vortex in that limit by choosing a contour containing eight sites $[(-1,-1),(-1,0),(-1,1),(0,1),(1,1),(1,0),(1,-1),(0,-1)]$, where the corresponding solution $e^{i\theta_{n,m}}$ on these sites, while it is 0 elsewhere (selecting without loss of generality $\mu=1$). The motivation for the choice of the phases over this discrete contour is that we aim to construct a solution with $S=2$ over the relevant contour, hence the real part of the configuration should behave as $\cos(2\theta)$, and the imaginary part as $\sin(2\theta)$, which in turn immediately implies that we should choose $\theta_{n,m}=j\pi/2$, where $j=1,\ldots,8$ is an index over contour sites. Below, we briefly discuss the general theory that would apply to any solution over the relevant contour, and then focus on the discrete vortex with $S=2$. The stationary state equation for $\phi_{n,m}$ is given by

$$f(\phi_{n,m},\bar{\phi}_{n,m},\epsilon)=(1-|\phi_{n,m}|^2)\phi_{n,m}-\epsilon(\Delta_2+4)\phi_{n,m}, \quad (2)$$

and its complex conjugate $\bar{f}(\phi_{n,m},\bar{\phi}_{n,m},\epsilon)=0$. The linearization operator for these two difference equations reads

$$\mathcal{H}_{n,m}=\begin{pmatrix} 1-2|\phi_{n,m}|^2 & -\phi_{n,m}^2 \\ -\bar{\phi}_{n,m}^2 & 1-2|\phi_{n,m}|^2 \end{pmatrix} - \epsilon(s_{+1,0}+s_{-1,0}+s_{0,+1}+s_{0,-1})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3)$$

with $s_{n',m'}u_{n,m}=u_{n+n',m+m'}$. Then, the solvability condition of the Lyapunov-Schmidt theory (allowing to continue a solution valid for $\epsilon=0$ to $\epsilon\neq 0$) mandates that the projection of the eigenvectors of $\mathcal{H}_{n,m}^{\epsilon=0}$ to Eq. (2) and its conjugate is null. To $O(\epsilon)$, this condition provides the bifurcation function constraint $g_j^1=\sin(\theta_j-\theta_{j+1})+\sin(\theta_j-\theta_{j-1})=0$, (for $j=1,\ldots,8$ with periodic boundary conditions) that was algebraically obtained in Ref. [23]. This is naturally satisfied for the $S=2$ vortex with $\theta_j-\theta_{j-1}=\pi/2$ discussed above. However, computing the Jacobian matrix $(\mathcal{M}_1)_{j,k}\equiv\partial g_j^1/\partial\theta_k$ of the bifurcation function g^1, one can observe that its eigenvalues are 0 for the case of our $S=2$ solution and hence second-order reductions are necessary to adjudicate on the existence and stability of the $S=2$ vortex.

Expanding $\phi_{n,m}=\phi_{n,m}^0+\epsilon\phi_{n,m}^1+O(\epsilon^2)$, one can obtain the corresponding equations for the $O(\epsilon)$ correction to the solution profile $\phi_{n,m}^0$ as

$$(1-2|\phi_{n,m}^0|^2)\phi_{n,m}^1-(\phi_{n,m}^0)^2\bar{\phi}_{n,m}^1=(\Delta_2+4)\phi_{n,m}^0. \quad (4)$$

The solution of Eq. (4) can be found as

$$\phi_{n,m}^1=-\frac{1}{2}\left[\cos(\theta_{j-1}-\theta_j)+\cos(\theta_{j+1}-\theta_j)\right]e^{i\theta_j}, \quad (5)$$

over the discrete contour while $\phi_{0,0}^1=e^{i\theta_2}+e^{i\theta_4}+e^{i\theta_6}+e^{i\theta_8}$. Using ϕ^1 to obtain the next order correction of the bifurcation function, we get

$$\begin{aligned} g_j^2=&\frac{1}{2}\sin(\theta_{j+1}-\theta_j)\left[\cos(\theta_j-\theta_{j+1})+\cos(\theta_{j+2}-\theta_{j+1})\right] \\ &+\frac{1}{2}\sin(\theta_{j-1}-\theta_j)\left[\cos(\theta_j-\theta_{j-1})+\cos(\theta_{j-2}-\theta_{j-1})\right] \\ &+(\sin(\theta_j-\theta_{j+2})+\sin(\theta_j-\theta_{j+4})+\sin(\theta_j-\theta_{j-2})) \\ &\times(\delta_{j,2}+\delta_{j,4}+\delta_{j,6}+\delta_{j,8}), \end{aligned}$$

with $1\leqslant j\leqslant 8$ and δ denoting the Kronecker symbol. Once again the bifurcation condition is satisfied for our vortex of $S=2$. However, the eigenvalues of the corresponding second-order Jacobian $\mathcal{M}_2$ are not identically zero and can be used to establish (in conjunction with the bifurcation condition being identically satisfied) the persistence of the vortex of topological charge $S=2$ in the vicinity of $\epsilon=0$.

Furthermore, the Jacobian $\mathcal{M}_2$ of the second order reductions can be computed explicitly as

$$\mathcal{M}_2=\begin{pmatrix} 1 & 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & -1 & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & 1 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & -1 \\ 0 & 0 & -\frac{1}{2} & 0 & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & -1 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & -1 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}. \quad (6)$$

By using the expansion

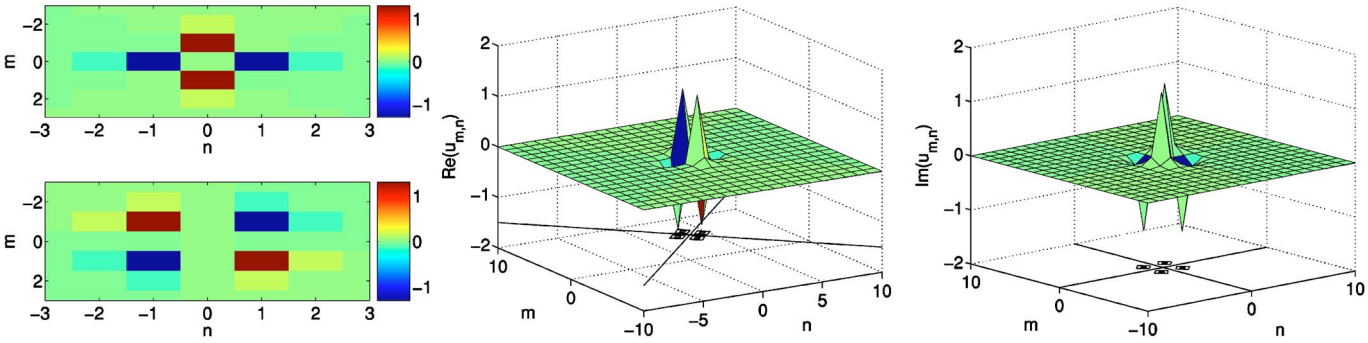

FIG. 1. (Color online) The discrete vortex is shown for $\epsilon=0.2$. The left panel shows contour plots of the real (top) and imaginary (bottom) parts. The middle panel shows a three-dimensional (3D) rendering of the real part, while the right panel shows a similar 3D plot of the imaginary part of the vortex.

$$u_{n,m}(t) = e^{i(1-4\epsilon)t+i\theta_0}\left(\phi_{n,m} + a_{n,m}e^{\lambda t} + \bar{b}_{n,m}e^{\bar{\lambda}t}\right), \tag{7}$$

one can study the stability of the discrete vortex of $S=2$. Furthermore, by expanding the eigenfunction in Taylor series in ϵ (as we did above for the solution ϕ) and correspondinglythe eigenvalue λ as $\lambda=\epsilon\lambda_1+O(\epsilon^2)$, it can be shown (see Ref. [20] for details) that the Jacobian $\mathcal{M}_2$ can be directly connected with the eigenvalue correction λ_1 (for eigenvalues bifurcating from 0, which are the natural sources of potential instability in the DNLS problem). The relevant equation connecting $\mathcal{M}_2$ and λ_1 is the (reduced, i.e., 8×8) eigenvalue problem of the form

$$\mathcal{M}_2\mathbf{c} = \lambda_1\mathcal{L}_2\mathbf{c} + \frac{\lambda_1^2}{2}\mathbf{c}, \tag{8}$$

where $(\mathcal{L}_2)_{j,k}=1$ if $j=k-1,-1$ if $j=k+1$, and 0 if $|j-k|\neq 1$. Using the discrete Fourier transform, one can obtain from Eq. (8) the characteristic polynomial:

$$\lambda_1^4 - 2\lambda_1^2\left(1-(-1)^j - 8\sin^2\frac{\pi j}{4}\right) + 8\sin^2\left(\frac{\pi j}{4}\right) \times\left(1-(-1)^j - 2\sin^2\frac{\pi j}{4}\right) = 0, \quad j=1,2,3,4, \tag{9}$$

which provides the leading order approximations to the eigenvalues of the $S=2$ vortex as follows: in the neighborhood of $\lambda=0$, the vortex will have three eigenvalues of algebraic multiplicity 4: $\lambda=0$ and $\lambda=\pm\sqrt{2}\epsilon i$, while it will have two simple imaginary eigenvalues $\lambda=\pm\sqrt{\sqrt{80}+8}\epsilon i$ and two real eigenvalues $\lambda=\pm\sqrt{\sqrt{80}-8}\epsilon$. Among the latter, the positive one is the reason for the $S=2$ vortex being *always* (i.e., for any $\epsilon\neq 0$) unstable, as was numerically observed [6].

The examination of the real eigenmode leading to the direct instability of the $S=2$ vortex [that has support over the central, i.e., (0,0), site], as well as the apparent mediation (in numerical experiments—see below) of the instability by means of the central site, lead us to consider the possibility of having an "impurity" at the central site, e.g., a strong localized potential such as a laser beam in BECs or an inhomogeneity in the photorefractive crystal, enforcing $\phi_{0,0}=0$. In such a case, the bifurcation function g_j^2 lacks the last term (encompassing the Kronecker symbols), since these are interactions "mediated" by the now inert (0,0) site. Furthermore, the second order Jacobian is now much simpler and acquires the form $(\mathcal{M}_2)_{j,k}=1$ for $j=k,-1/2$ for $j=k\pm 2$, and 0 for $|j-k|\neq 0,2$. One can then repeat the calculation of the eigenvalues in the problem of Eq. (8), via the discrete Fourier transform, to obtain the characteristic equation

$$\left(\lambda_1 + 2i\sin\left(\frac{j\pi}{4}\right)\right)^2 = 0, \quad j=1,\ldots,8. \tag{10}$$

This results in three eigenvalues of algebraic multiplicity 4, namely $\lambda=0$ and $\lambda=\pm\epsilon i\sqrt{2}$. There are also two double eigenvalues, $\lambda=\pm 2i$. The crucial observation, however, is that in this case, there are no real eigenvalues immediately present as $\epsilon\neq 0$ and hence the discrete vortex with $S=2$ will be *linearly stable*, due to the stabilizing effect of the impurity (or, to be more precise, due to the absence of the instability mediated by the (0,0) site). We now turn to numerical investigations to examine the validity of these findings.

III. NUMERICAL RESULTS

We identify unit frequency solutions with topological charge $S=2$, by initializing the exact solution configuration at the $\epsilon=0$ limit of Eq. (1) and then using continuation over ϵ, combined with a contraction mapping for the solution of the nonlinear system of (algebraic) equations to identify the exact (up to a prescribed numerical accuracy) numerical discrete vortex. We then perform linear stability analysis, using the expansion of Eq. (7), to obtain the eigenvalues λ, and their corresponding eigenvectors.

Figure 1 shows a typical example of the discrete vortex for $\epsilon=0.2$ in the regular DNLS model. The middle and right panels show the real and imaginary part of the solution, clearly emulating $\cos(2\theta)$ and $\sin(2\theta)$ over the lattice contour of interest. The linear stability analysis of this vortex is shown in Fig. 2. One can observe that both for the imaginary (top left panel), as well as for the real (bottom left panel) eigenvalues, the predictions of the perturbation theory (dashed line) are extremely accurate in comparison with the full numerical results even for ϵ up to 0.25. Clearly as ϵ increases, higher order phenomena become relevant, such as the splitting of the quartet of eigenvalues at $\lambda=\pm\sqrt{2}\epsilon i$,

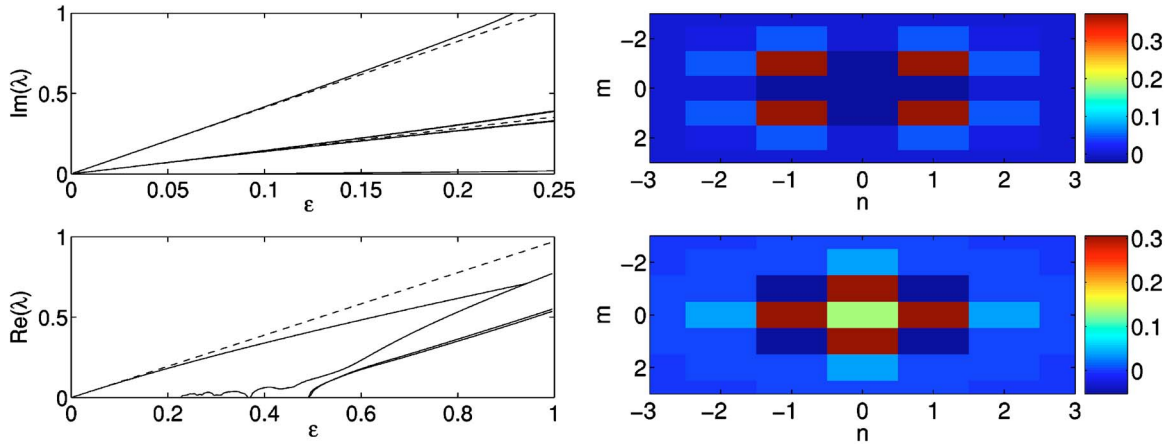

FIG. 2. (Color online) Linear stability analysis of the $S=2$ vortex. The left panel shows the imaginary (top) and the real (bottom) part of the point spectrum eigenvalues bifurcating from $\lambda=0$. The solid line shows the numerical result, while the dashed line indicates the theoretical prediction (see text). The most unstable eigenmode pertaining to the real eigenvalue for $\epsilon=0.2$ is shown (real part: top; imaginary part: bottom) in the right panel.

or the collision of the simple pair of eigenvalues with $\lambda=\pm\sqrt{\sqrt{80}+8\epsilon}i$ with the bottom edge of the continuous spectrum (which is at $\lambda=\pm i$), resulting in a Hamiltonian Hopf bifurcation and a complex quartet of eigenvalues for $\epsilon>0.23$. Additional such quartets appear for larger values of ϵ. However, the solution is always unstable due the real eigenvalue pair $\lambda=\pm\sqrt{\sqrt{80}-8\epsilon}$, whose eigenfunction is shown in the right panel of the figure. Notice that the latter has support over the central site, predisposing us for the role of this site in the instability development.

The corresponding predictions/numerical results for the model with the impurity [i.e., with (0,0) inert] are shown in Fig. 3. The top panel illustrates the eigenvalue of multiplicity four with $\lambda=\pm\epsilon i\sqrt{2}$ and with multiplicity two $\lambda=\pm 2i$, which are again in excellent agreement with the numerical findings. The real part clearly indicates the *absence* of an instability for small ϵ. Such an instability arises due to collision of the eigenvalue pair with the continuous spectrum and is present for $\epsilon>0.36$. It is crucial to note here that the $S=1$ vortex was found to be stable for $\epsilon<0.38$ in Ref. [20]. This illustrates that the present mechanism stabilizes the $S=2$ vortex for a parametric region comparable to that of the $S=1$ vortex, hinting that it could be experimentally feasible to trace such a configuration similarly to what was done for the $S=1$ case [8]. The right panel shows the real and imaginary part of an unstable eigenmode for $\epsilon=0.4$.

Finally, to examine the dynamical development of the instability and to compare and contrast the dynamical features of the two models (in the absence and presence, respectively, of the impurity), we have conducted direct numerical experiments. The main results are shown in Fig. 4 for two representative cases (namely $\epsilon=0.2$, where the former case is unstable, while the latter is stable, and $\epsilon=0.4$, where both models have unstable $S=2$ vortices). In both cases, we have simulated both models up to $t=200$, initializing them with *identical* initial conditions consisting of the vortex with a perturbation (multiplied by 10^{-4}) in the eigendirection of the right panel of Fig. 2 for $\epsilon=0.2$ and of Fig. 3 for $\epsilon=0.4$. In the case of $\epsilon=0.2$, we observe that the DNLS vortex becomes unstable, whereas in the presence of the impurity the solution is completely stable (exhibiting oscillations at the order of the initial perturbation). The instability for the DNLS vortex appears to be mediated by the central site (dash-dotted) line,

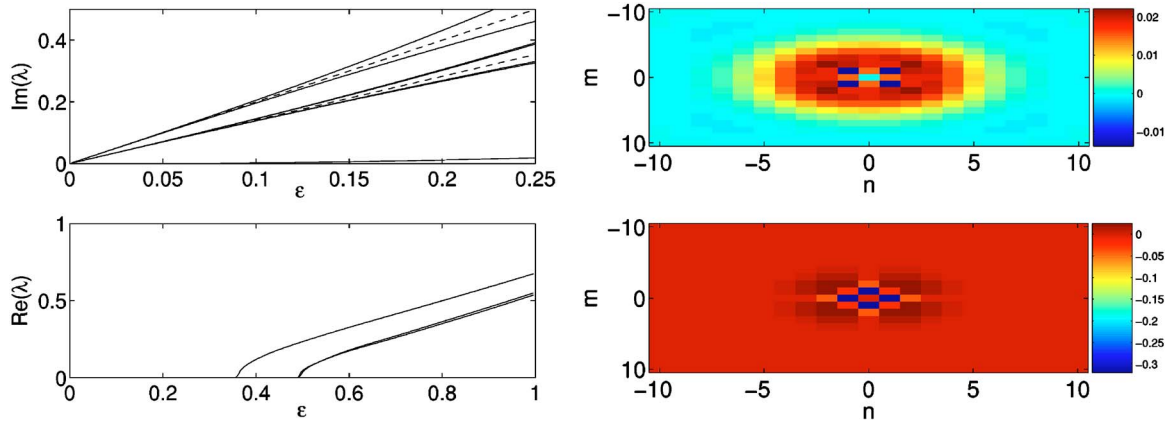

FIG. 3. (Color online) Same as in Fig. 2, but for the model with the impurity [i.e., with (0,0) inert]. The instability is absent in this case. The right panel shows the principal eigenmode of instability for $\epsilon=0.4$.

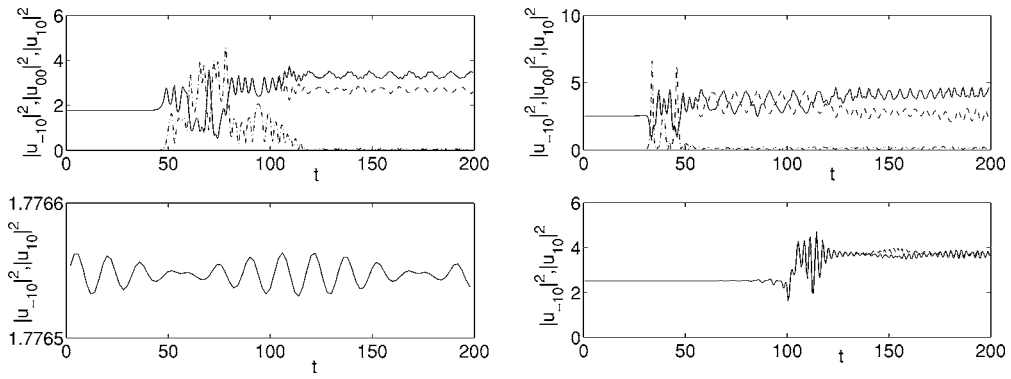

FIG. 4. For identical initial conditions (see text), $\epsilon=0.2$ (left) and $\epsilon=0.4$ (right), the solution is shown in the relevant central sites as a function of time, in the absence (top) and presence (bottom) of the impurity. The central site (dash-dotted line) and the sites $(-1,0)$ (solid line) and (1, 0) (dashed line) are shown (the latter two are the only ones that remain excited in the configuration for long times).

which eventually settles at a rather small amplitude. The final configuration finds six of the eight (initially) participating sites at the vortex with near-zero amplitudes, while only two sites (shown by solid and dashed line) remain excited in an asymmetric configuration with a long-lived breathing (weak) exchange of power between them, mediated principally by the central site. The instability sets in for $t\approx 45$ in this case. For $\epsilon=0.4$, the regular DNLS becomes unstable even faster ($t\approx 30$) and the dynamics is similar. For the case with the impurity the instability sets in at much longer times ($t\approx 80$), as expected by the much smaller value of the corresponding principal eigenvalue real part. Furthermore, while only two sites remain excited in this case as well, the configuration is no longer asymmetric and the oscillation of power can be identified (data not shown) as being caused by the small amplitude exchange of power around the vortex contour (recall that the central site is in this case inert).

IV. CONCLUSIONS

In this work we have revisited the topic of discrete vortices of topological charge $S=2$. We have explicitly discussed and illustrated their instability in the prototypical lattice model of the discrete nonlinear Schrödinger equation and have traced its source in the exchange of power made available through the central site of the vortex. We have thus proposed to consider a model with an impurity (an inert) site at the center of the vortex. Examination of the stability problem in the latter case shows the absence of linear instability for a regime of coupling strengths comparable to that of the linear stability interval of the experimentally observable $S=1$ state. Numerical findings conclusively corroborate this picture both at the level of linear stability analysis (found to be in excellent agreement with the theoretical predictions) and at the one of direct numerical experiments. We believe that this opens the path for observation of higher charge discrete vortices and renders experimental work in this direction particularly timely.

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